THE GRAPHS OF POLYTOPES WITH INVOLUTORY AUTOMORPHISMS

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ABSTRACT

Steinitz' theorem states that a graph is the graph of a 3-dimensional convex polytope if and only if it is planar and 3-connected. Grünbaum has shown that Steinitz' proof can be modified to characterize the graphs of polytopes that are centrally symmetric or have a plane of symmetry. We show how to modify Steinitz' proof to take care of the remaining involutory case—polytopes that are symmetric about a line.

1. Introduction

One of the most important theorems in the study of convex polytopes is Steinitz's Theorem which states that *a graph is the graph of a 3-dimensional polytope (hereafter to be called a 3-polytope) if and only if it is planar and 3-connected.* Grünbaum has modified one of Steinitz's proofs of this theorem to characterize the graphs of polytopes with a center or plane of symmetry [1]. In this paper we shall prove

Theorem 1. *If G is a planar 3-connected graph with a non-trivial involutory automorphism ϕ which preserves orientation, then G is the graph of a polytope P with a line of symmetry. Moreover, the automorphism ϕ is induced on the graph of P by this symmetry.* (By a *line of symmetry* we mean a line L such that P is a reflection of itself through L).

Combining Theorem 1 with Grünbaum's results (see Remark 2 at the end of this paper) we have as a corollary:

Any non-trivial involutory automorphism of a planar 3-connected graph G is one of those corresponding to central symmetry, a plane of symmetry or a line of symmetry of a polytope whose graph is G.

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Our proof of Theorem 1 is also a modification of Steinitz's proof. Since an exposition of his proof would be quite lengthy, we recommend that the reader read [1, pp. 235–253] in order to understand our methods and terminology.

2. Definitions

Two graphs G and H are *isomorphic* provided there is a 1-1 map ϕ of the vertices of G onto the vertices of H so that $v_1 v_2$ is an edge of G if and only if $\phi(v_1)\phi(v_2)$ is an edge of H (the function ϕ will be called an *isomorphism*). We shall say that two 3-polytopes are *isomorphic* provided their graphs are isomorphic. An *automorphism* of G is an isomorphism of G onto G. An automorphism ϕ is *involutory* provided ϕ composed with itself is the identity map, and ϕ is *nontrivial* provided it is not the identity map.

A *face* of a planar 3-connected graph G embedded in the plane π is a circuit bounding a component of $\pi \sim G$. This definition does not depend on the embedding, and the faces of G correspond to the faces of any 3-polytope P whose graph is G. By an *element* of G (or P), we mean either a vertex, edge or face of G (or P). A point p is *beyond* a face F of a polytope P if the plane determined by F separates p from P; it is *beneath* F if p and P lie on the same side of this plane.

3. Proof of Theorem 1

Our proof is by induction on the number of edges of G with edges fixed by ϕ being counted twice. We apply one of the reductions used by Steinitz, [1, p. 238, Fig. 13.1.2] to G to produce a graph G' with fewer edges. From a polytope P' whose graph is G' and which has a line of symmetry we construct a polytope P whose graph is G and which also has a line of symmetry.

If we cannot apply a reduction which decreases the number of edges we shall use a reduction which will reduce the number of faces in some lens of $\mathscr{I}(G)$. After a finite number of such reductions we will again have a graph for which a reduction will reduce the number of edges.

In order to have a graph with an involutory automorphism after applying a reduction, we shall apply reductions in pairs. If A is a face or vertex of G to which we wish to apply a reduction then we shall also apply the reduction to $\phi(A)$. Our proof parallels the proof for central symmetry and a plane of symmetry [see 1] except that certain complications may arise. In the following we shall show what these complications are and how to modify the proof in [1] to take care of them.

COMPLICATION 1. It may be that after we have applied ω_i to v we have altered G so that ω_i cannot be applied to $\phi(v)$. This can happen only when $v\phi(v)$ is an edge, in which case we modify ω_i as illustrated in Fig. 1.

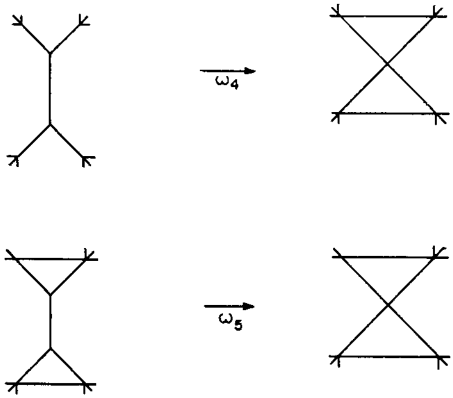

Fig. 1

Now that we have introduced new reductions we must check to see that they do what they are supposed to. The reduction ω_5 reduces the number of edges in G and may be used in our induction. The reduction ω_4 leaves the number of edges unchanged (since $v\phi(v)$ is counted twice). This is no problem because we use ω_4 to replace ω_0, thus what we require of ω_4 is that it reduce the number of faces in a lens of $\mathscr{I}(G)$ when applied judiciously.

Suppose L is the lens we want to reduce by applying ω_4 to G, and let T be the triangular face of L corresponding to v. If T does not meet a pole of L then the application of ω_4 will destroy L (see Fig. 2).

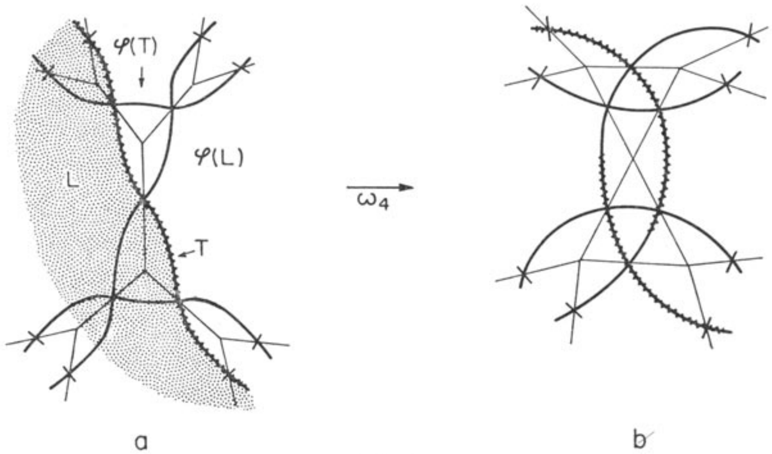

Fig. 2

To get around this problem we note that in [1, p. 241] it is proved that a lens with interior vertices has a triangular face meeting one of the bounding geodesics. Essentially the same proof shows that there are two such faces, one meeting each bounding geodesic. Let T' be another triangular face in L, meeting the other bounding geodesic. If we cannot apply ω_4 to the element of G corresponding to T' then we have that T' and $\phi(T')$ are situated as in Fig. 2a. This implies that geodesics bounding $\phi(L)$ are also the geodesics bounding L. This can happen only if $L = \phi(L)$ which contradicts the fact that $\phi(T)$ is not in L.

Next we check the case where T meets a pole. By drawing a sketch of this situation one may easily verify that ω_4 will decrease the number of faces of L.

COMPLICATION 2. Even though ω_0 or η_0 will decrease the number of faces of L, after a pair of reductions have been applied the net effect may be no change in the number of faces, or a decrease of at least two, destroying the lens.

We consider first the case where the lens is destroyed. This will happen only if L is the lens $\mathscr{L}$ of Fig. 3a.

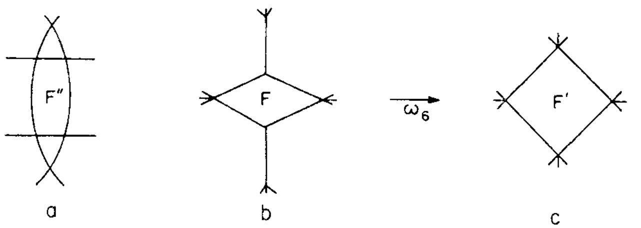

Fig. 3

Corresponding to $\mathscr{L}$ will be the configuration in Fig. 3b (or its dual) in G. We shall use a reduction ω_6 which will change this configuration to the one in Fig. 3c (or its dual). We shall use this reduction only for the purpose of obtaining more information about G. This reduction decreases the number of edges and thus, by induction, the new graph G' is the graph of a 3-polytope P' with a line of symmetry and the automorphism ϕ' induced in G' by ϕ is also induced in G' by the symmetry of P'. We may conclude that at most two elements of G are fixed by ϕ' and thus at most two elements of $\mathscr{I}(G)$ are fixed by ϕ. We may use the same argument for the case where the dual configuration is in G.

It is possible that ω_6 does not preserve 3-connectedness. One may check that in this case ω_7 (see Fig. 4) will preserve 3-connectedness and the same argument will hold.

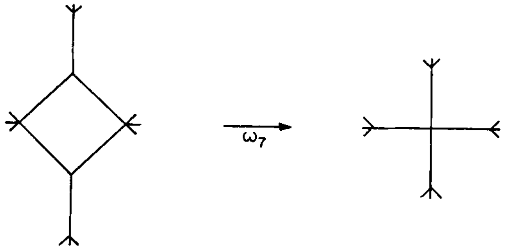

Fig. 4

The lens $\mathscr{L}$ gives us trouble only if $\phi(\mathscr{L}) = \mathscr{L}$, for otherwise we apply our reduction to only one triangular face of $\mathscr{L}$. This shows that a "bad" lens of type $\mathscr{L}$ has its quadrilateral face fixed by ϕ.

We shall show that some other lens can be reduced so that eventually the number of faces in a minimal irreducible lens is reduced.

Case I. All geodesics in $\mathscr{I}(G)$ are simple. In this case any two intersecting geodesics will determine at least four lenses which have disjoint interiors. At least one will contain an irreducible lens, $\mathscr{L}'$, that has no element in its interior that is fixed by ϕ. Whenever a reduction is applied to $\mathscr{L}'$, this property will be preserved, and so in applying reductions to $\mathscr{L}'$ we will never obtain the lens in Fig. 3a and we may apply reductions until the number of faces in $\mathscr{L}'$ is less than 3.

Case II. Some geodesic intersects itself. We choose a geodesic arc Γ such that:

1. The endpoints of Γ coincide, and Γ forms a simple closed curve.
2. If we extend Γ by an edge at each end then these edges lie outside Γ.
3. Γ is minimal in the respect that no geodesic intersects itself inside Γ.

Consider what happens now if we follow along the geodesic determined by Γ, proceeding along a portion of this geodesic starting with an edge meeting Γ and lying outside Γ. This path, which we shall call P_1, can do one of three things: it may return to some vertex of Γ other than the vertex v of self intersection, it may return to v, or it may intersect itself. If neither of the last two cases happen, then we may consider a similar path P_2 from the other edge meeting v. If neither of the latter two cases happen to P_2 then either P_2 returns to Γ or to P_1, producing at least three lenses with disjoint interiors. As in case one, we may choose one of these and apply reduction to some irreducible lens in it.

If either of the latter two cases does happen then there is a geodesic arc outside Γ satisfying 1 and 2 and thus we may choose a geodesic arc Γ_1 outside Γ satisfying 1, 2 and 3.

Any geodesic crossing Γ_1 (or Γ_2) will form a lens with a portion of Γ_1 (or Γ_2). We have seen that G contains at most two elements that are fixed by ϕ, and it is the presence of such elements in irreducible lenses that cause us trouble. Thus we can find a lens to reduce unless $\mathscr{L}$ is one of the irreducible lenses in a lens formed by cutting Γ_1 (or Γ_2) by a geodesic.

Let Γ_3 be a geodesic crossing Γ_1, forming a lens $\mathscr{L}_1$ that is minimal in the sense that it does not properly contain any such lens. We may assume without loss of generality that $\mathscr{L}$ is an irreducible lens of $\mathscr{L}_1$ meeting the boundary of $\mathscr{L}_1$, and that the other element fixed by ϕ lies inside Γ_2. If $\mathscr{L}_1 = \mathscr{L}$ then we choose a geodesic Γ_4 determined by an edge crossing $\mathscr{L}$. This geodesic will form a lens $\mathscr{L}_2$ with a portion of Γ_1, and $\mathscr{L}_2$ will be one of two types: either $\mathscr{L}_2$ will not contain the 4-sided face of F of $\mathscr{L}$ or it will contain F, and F will be incident to a pole of $\mathscr{L}_2$. In either case applying reductions to $\mathscr{L}_2$ will never produce a lens of type $\mathscr{L}$.

If $\mathscr{L}_1 \neq \mathscr{L}$ then $\mathscr{L}$ has a portion of Γ_3 on its boundary and a portion of another geodesic Γ_5 also on its boundary. There are two ways in which Γ_5 may intersect Γ (See Fig. 5). In Fig. 5a, Γ_5 forms a lens with a portion of Γ and this lens does not contain a fixed element and we are done. In Fig. 5b we consider a geodesic Γ_6 determined by an edge crossing $\mathscr{L}$, as illustrated in Fig. 5b. If Γ_6 crosses Γ_5 only once inside Γ then Γ_6 and Γ will form a lens without any fixed elements. If Γ_6 crosses Γ_5 again inside Γ then Γ_6 and Γ_5 form a lens that either does not contain F or has F incident to a pole. In either case we have found a lens that we can reduce.

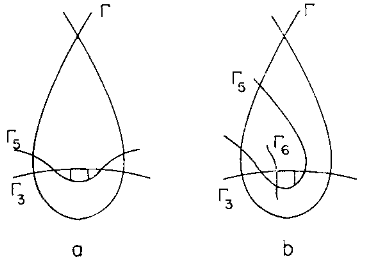

Fig. 5

Proceeding with our investigation of Complication 2, we turn to the possibility that a pair of reductions leaves the number of faces in our lens unchanged. There are two cases.

Case I. The lens L, which we wish to reduce, and $\phi(L)$ meet at their poles.

Rather than repeating the argument taking care of this case we refer the reader to [1, p. 246].

Case II. The lenses L and $\phi(L)$ meet along a bounding geodesic. We refer the reader to Complication 1 where just such a case was disposed of.

We are now ready to take the polytope P' whose graph was obtained by reducing G, and which has a line of symmetry, and construct from it a polytope P with a line of symmetry and with G as its graph. Here we encounter two more complications.

COMPLICATION 3. In applying ω_0^{-1} to a polytope it is sometimes necessary to apply a projective transformation. This we wish to avoid because we want to preserve symmetry at each step. We shall show that in most cases a projective transformation is not necessary.

If F and $\phi(F)$ are the faces to which we wish to apply ω_0^{-1} then what we need is that the planes π_1, π_2 and π_3 determined by the faces that abut F (i.e. meet F on an edge) meet at a point p beyond F. Suppose p lies beneath F. The polytope P' lies inside the polytope $\mathrm{con}(\{p\} \cup F)$. If $P' \neq \mathrm{con}(\{p\} \cup F)$ then $\phi(F)$ is a subset of some cross section of $\mathrm{con}(\{p\} \cup F)$, moreover, this cross section misses F. This leads to a contradiction because the cross section has area less than the area of F but the area of F and $\phi(F)$ are equal.

Let $\mathscr{P} = \mathrm{con}(F \cup \phi(F))$. The angle between F and a face abutting F is no greater than the corresponding angle in P', thus if we can show that the planes determined by the faces abutting F in $\mathscr{P}$ meet at a point beyond F then we are done. By the above argument, these planes cannot meet at a point beneath F, unless $\mathscr{P}$ is a tetrahedron, so we shall assume they meet at infinity. In the half-space on the other side of the plane determined by F, from P' the planes π_4, π_5 and π_6 determined by faces abutting $\phi(F)$ lie on or outside the cylinder determined by faces abutting F. If this were not the case then three planes determined by faces of P would meet beyond F and thus π_1, π_2 and π_3 would meet beyond F. Since the cylinder determined by π_4, π_5 and π_6 is congruent to the cylinder determined by π_1, π_2 and π_3 the two must coincide.

The above allows us to conclude that we do not need a projective transformation unless $\mathscr{P}$ is a tetrahedron, a triangular prism or a quadrilateral pyramid. In the last two cases π_1, π_2 and π_3 meet at infinity and thus we do not need a projective transformation unless $P' = \mathscr{P}$. We will not apply a transformation to these two cases, we simply observe that if we apply ω_0^{-1} to the graphs of these polytopes we do not obtain a polyhedral graph, thus these two cases never arise.

If $\mathscr{P}$ is a tetrahedron then F abuts $\phi(F)$ and we may perform the following to construct the desired polytops:

Let F_1 and F_2 be the other faces abutting F in P'. Choose a point p_1 on the intersection of F_1 and F_2 and let p_2 be the reflection of p_1 through the line of symmetry. Among all segments $[p_1, p_2]$, one must intersect the edge common to F and $\phi(F)$. We take the convex hull of this segment with P and we have the polytope with the desired graph and symmetry.

COMPLICATION 4. When we perform the operations $\bar{\omega}_i^1$ to P', what we do is to choose a point beyond a certain face F with the property that it is on a specified set of planes determined by planes abutting F. It is possible that after performing ω_i^{-1} to F we have altered the faces abutting $\phi(F)$ so that a second point chosen symmetrically will not lie on the proper planes. This can happen only if F abuts $\phi(F)$ in which case easy variants of the construction at the end of Complication 3 will serve our purpose.

It remains now only to start the induction. Here is where we take care of

COMPLICATION 5. After applying one of a pair of reductions we may have reduced G to the graph of the tetrahedron. This will happen when G is the graph of the bipyramid over a triangle, the pyramid over a quadrilateral or the triangular prism. The theorem is easily verified in these cases and they, together with the tetrahedron, serve to start the induction.

REMARKS

1. Due to limitations of space. Grünbaum gave only outlines of the proofs for the cases of a center or plane of symmetry. This paper should give the reader some idea of what is involved in filling in the details.

2. Grünbaum has informed the author that the statement of Theorem 6 [1, p. 246] should read:

A graph $\mathscr{G}$ is realizable by a 3-polytope having a plane of symmetry if and only if $\mathscr{G}$ is planar and 3-connected, and there exists an involutory mapping ϕ of $\mathscr{G}$ which reverses orientation of the faces and fixes at least one edge or vertex.

3. The author has convinced himself that the following can be proved using methods similar to those in this paper.

If P_1 and P_2 are two isomorphic 3-polytopes with a center (line) of symmetry and if P_1 and P_2 have the same orientation then P_1 can be continuously deformed into P_2 so that all intermediate polytopes are isomorphic to P_1 and have a center (line) of symmetry.

The author conjectures that the same is true for polytopes with a plane of symmetry.

References

1. B. Grünbaum, *Convex Polytopes*, Wiley and Sons, New York, 1967.
2. E. Steinitz and H. Rademacher, *Vorlesungen über die Theorie der Polyeder*, Berlin, 1934.

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